Tolman-Bondi model, fractal density and Hubble law. 1. Initial conditions.

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Abstract.

Properties of Tolman-Bondi (TB) model produced by two set of initial conditions, 1) fractal density and simultaneous bang time and 2) fractal density and linear Hubble law, are studied. It is shown for the first set that for some physical resonable values of parameters of the model, the central density and the cosmological density parameter Ω_0, an area of compatibility of initial conditions has the form $\xi > \xi_{TB}$, where ξ is radial Euler coordinate and ξ_{TB} is the low limit of the area, where particle has zero velocity. For the second set of initial conditions it is shown that the area of compartibility is trivial, $\xi \geq 0$ only for non-simultaneous bang time.

A case of an arbitrary bang time is also studied.

It is shown that in the frame of the exact nonliner relativistic TB models it is possible to have a linear velocity - distance relation of the expanding space when matter distribution is fractal. This requires a non-unique bang time. The bang time $\tau(\xi)$ is calculated for the linear Hubble law and fractal matter distribution with fractal dimension $D = 2$.

1. Introduction

The Tolman-Bondi (TB) models are exact nonlinear solutions of Einstein's equations under the assumptions of 1) spherical symmetry, 2) pressureless matter (dust) and 3) motion with no particle layers intersecting. Originally studied by Lemaitre (1933), Tolman (1934) and Bondi (1947), these models are the simplest generalization of the Friedmann-Robertson-Walker (FRW) models with a non-zero density gradient.

At least two important cosmological applications of TB models have recently been discussed in the literature. The first one is related to the evolution of primordial inhomogeneities in an expanding universe Silk & Wilson (1979a), Olson & Silk (1979), Silk & Wilson (1979b), Olson & Stricland (1990), Teerikorpi *et al.* (1992), Ekholm & Teerikorpi (1993). In particular, very inportant results using TB models have been obtained under the assumption of "no bang time variation" (Olson & Silk 1979) or of "unique bang time", i.e. when there is a simultaneous creation time for every mass shell. For instance, in references Silk & Wilson (1979a), Olson & Silk (1979), Silk & Wilson (1979b), Olson & Stricland (1990), the formation of galactic clusters from small density and velocity perturbations was studied (implicitly, non-simultaneous bang is used in Silk ans Wilson (1979b)), and it was shown that at after a sufficiently large time the initial conditions are forgotten and a universal density profile is formed. In reference Olson & Silk (1979), two theorems were proved about the development of halos of excess density around spherical galaxy clusters, also based on this assumption. In the present paper we show that the assumption of a unique bang time implies a strong restriction on the parameter domain where the TB models have a solution.

The second application area is the modeling of fractal matter distribution within general relativity Bonnor (1972), Ribeiro (1992a,1992b,1993), Humphreys *et al* (1998b), Matravers (1998). Modern redshift surveys of galaxies have revealed a fractal structure with the fractal dimension $D \approx 2$ in the space distribution of galaxies up to distances of $100h^{-1}$ Mpc ($h = \frac{H_0}{100km \cdot s^{-1} \cdot Mpc^{-1}}$, here H_0 is the Hubble constant) (see Silos Labini *et al* 1998). This has confirmed the scale invariant de Vaucouleurs (1970) law for galaxy distribution and leads to a new application of the TB models, as first pointed out by Bonnor (1972). In this application the fractal structure is treated as a spherically symmetrical inhomogeneity with a preferred center. Baryshev *et al* (1998) demonstrated that the linear perturbation approximation for the gravitational growth of spherical density fluctuations in the case of fractals leads to a non-linear Hubble law if all matter is included into fractals. Then the observed linear Hubble law (at scales less than $100h^{-1}$ Mpc) requires that the background (FRW) density is very low. The same conclusions were obtained with exact TB models calculations by Humphreys *et al* (1998b) and Matravers (1998). Baryshev *et al* (1998) proposed another solution for the paradox of the linear Hubble law within the fractal structure (the so-called Hubble-de Vaucouleurs paradox): homogeneously distributed dark matter with a very high density. † In the

† In an important earlier paper of 1972, Sandage *et al* (1972) were perhaps the first to note the surprising co-existence the linear Hubble law and the *local* inhomogeneities. Though, they concluded on the basis of the galaxy counts available at that time that the space number density does not decrease around us as predicted by the de Vaucouleurs law. Recently, however, Teerikorpi *et al* (1992), have shown, using a new method based on photometric Tully-Fisher diskances, that the all-sky average number density decreasies as predicted by the fractal dimension ≈ 2, from 1 to $100h^{-1}Mpc$.

present paper we show that there is still a third way to make the linear Hubble law, by abandoning the assumption of a unique bang time.

Before demonstrating this, we note that there are two ways to parameterize Tolman-Bondi models. The first, introduced by Tolman and Bondi, has been called $3+1$ approach (see discussion e.g., Matravers, 1998). The second one uses observational coordinates (Ellis, Nel et al, 1985). In case of small scales, as in individual galaxy clusters, the difference between these approaches is negligible. It is necessary to use observational coordinates, which utilize the past light cone of an observer, when one discusses observation at large redshifts. As underlined by Matravers (1998), the $3+1$ coordinate approach provides a physical interpretation of the evolution of the universe in co-moving coordinates and it is accepted in the present paper as a first step of investigation.

The present paper studies the initial conditions of the TB models which are conserning to the present observations. In the second paper we will study dynamics, produced by these initial conditions (Gromov *et al* 1999).

In section 2 we review the TB models and discuss two approaches: $3+1$ and observational coordinates approach. In section 3 we introduce and study the domain of definition of TB model and formulate two forms of criterium for checking if the domain is not trivial. In section 4 we show the predicted radial velocity deflection from the Hubble law within TB models with unique bang times and density distribution with fractal dimension $D = 2$; apply the criterium of domain definition of TB model and show (Tables 1 - 4) how the domain depends on initial conditions and cosmological density parameter Ω_0. We calculate also the bang time which is able to reproduce at the present cosmic epoch the linear Hubble law within the high density inhomogeneities described by TB models.

2. A review of the TB models

In this section we review two basic representations of the TB models: TB models in co-moving coordinates and TB models in observational coordinates.

2.1. TB models in co-moving coordinates

The TB models are the simplest exact, nonlinear, inhomogeneous, nonstationary, spherically symmetrical dust models in general relativity. The models are formulated in co-moving (Lagrangian) and synchronous coordinates r, t in the metric

$$\mathrm{d}s^2(r,t) = c^2\,\mathrm{d}t^2 - \mathrm{e}^{\lambda(r,t)}\,\mathrm{d}r^2 - R^2(r,t)\mathrm{d}\Omega^2 \tag{1}$$

for a stress-energy tensor in the form

$$T_0^0 = \rho\, c^2(r,t) \qquad T_i^k = 0 \qquad (i \neq 0, k \neq 0)\,, \tag{2}$$

where c is the speed of light, $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin\theta\,\mathrm{d}\phi^2$, $R(r,t)$ is an Euler coordinate. The class of metrics given by (1) together with the stress-energy tensor (2) produce a set of inhomogeneous cosmological models, generally with time- and space-dependent curvature, for which the Einstein's equations are reduced to the following system (Tolman 1934):

$$\frac{8\,\pi\,G}{c^4}\,T^0_0 = -\frac{\mathrm{e}^{-\lambda}}{R^2}\,\left(2\,R\,R'' + R'^{\,2} - R\,R'\,\lambda'\right) + \frac{1}{R^2}\,\left(R\,\dot{R}\,\dot{\lambda} + \dot{R}^2 + 1\right), \quad (3)$$

$$\frac{8\,\pi\,G}{c^4}\,T^1_1 = -\mathrm{e}^{-\lambda}\,R'^{\,2} \;+ 2\,R\,\ddot{R} + \dot{R}^2 + 1 = 0, \quad (4)$$

$$\frac{8\,\pi\,G}{c^4}\,T^2_2 = -\frac{\mathrm{e}^{-\lambda}}{R}\,\left(R\,R'' - \frac{R'\,\lambda'}{2}\right) + \frac{\dot{R}\,\dot{\lambda}}{2\,R} + \frac{\ddot{\lambda}}{2} + \frac{\dot{\lambda}^2}{4} + \frac{\ddot{R}}{R} = 0, \quad (5)$$

$$T^1_0 = 2\,\dot{R}' - \dot{\lambda}\,R' = 0, \quad (6)$$

$$T^2_2 = T^3_3, \quad (7)$$

where $' = \frac{\partial}{\partial r}$ and $\dot{} = \frac{\partial}{\partial(c\,t)}$. A subset with space-constant curvature are the FRW models. In turn, the Newtonian theory arises as the limit of FRW models when we neglect the difference between the gravitational mass of the dust and the invariant mass.

Applying Einstein's equations to the equation (6), the function $\lambda(r,t)$ is defined as

$$\mathrm{e}^{\lambda(r,t)} = \frac{R'^{\,2}(r,t)}{f^2(r)}, \quad (8)$$

where $f(r)$ is one of the undetermined functions of the models.

Using (8), the equation (4) is reduced to the equation of motion

$$2\,\ddot{R}(r,t)R(r,t) + \dot{R}^2(r,t) + 1 - f^2(r) = 0 \quad (9)$$

with initial conditions:

$$R(r,t)|_{t=t_0} = R_0(r), \quad (10)$$

$$\dot{R}(r,t)\Big|_{t=t_0} = \dot{R}_0(r), \quad (11)$$

where initial conditions $R_0(r)$ and $\dot{R}_0(r)$ represent the values on the "now" hypersurface. The equation (3) for the density is:

$$\frac{8\,\pi\,G}{c^4}\,T^0_0(r,t) = \frac{\mathrm{d}F(r)}{\mathrm{d}r}\,\frac{1}{2\,R^2(r,t)\,\frac{\partial R(r,t)}{\partial r}}, \quad (12)$$

where $F(r)$ is a second undetermined function. As can be seen from (12), the models becomes singularitiy by two different couses. The first is defined by $R(r,\tau) = 0$, while the second one is defined by $R'(r,\tau)=0$. Two characteristic functions, the bang time (Silk & Wilson (1979a), Olson & Silk (1979), Silk & Wilson (1979b)), and layer intersection time function (see, for instance, Gromov 1997), respectively, correspond to these singularities.

A relationship involving the two undetermined functions can be found by considering the different types of mass used in the models. The total mass M_{grav} of the dust is defined by the stress-energy tensor (Landau & Lifshitz, 1973):

$$M_{grav}(R) = \frac{4\pi}{c^2}\int\limits_0^R T_0^0(x,t)\,x^2\,\mathrm{d}x, \tag{13}$$

while the invariant mass M_{inv} (Bondi, 1947):

$$M_{inv}(r) = \frac{4\,\pi}{c^2}\int\limits_0^r T_0^0(x,t)\,\sqrt{-g(x,t)}\mathrm{d}x, \qquad \sqrt{-g} = \frac{R'\,R^2}{f}. \tag{14}$$

Substituting ρ from (2) and (12) into (13) and (14) we obtain

$$M_{grav} = \frac{c^2}{4\,G}\,F(R), \qquad F(0)=0, \tag{15}$$

and

$$M_{inv}(R) = \int\limits_0^R \frac{\mathrm{d}M_{ADM(x)}}{f(x)}, \qquad f(R) = \frac{M'_{grav}}{M'_{inv}}. \tag{16}$$

The Euler coordinate R depends on time but M_{grav} and M_{inv} both are implicitly *not* time dependent. Bondi (1947) showed that there are two ways of interpreting the function $f(r)$. The first is related to (16). The second is that it is related to the curvature and components of the Einstein tensor (Bondi 1947):

$$K_1^1 = 2\,\frac{f}{R}\,\frac{\mathrm{d}f}{\mathrm{d}R} \qquad K_2^2 = K_3^3 = \frac{f^2-1}{R^2} + \frac{f}{R}\,\frac{\mathrm{d}f}{\mathrm{d}R}, \qquad \mathbf{K} = \frac{2}{R^2}\,\frac{\mathrm{d}}{\mathrm{d}R}\left(R\,\left(f^2-1\right)\right).$$

It is apparent that the space curvature is equal to zero if and only if $f=1$.

Returning to the equation of motion, we see that the first and second integrals of (9) are:

$$\frac{1}{2}\dot{R}^2(r,t) = \frac{f^2(r)-1}{2} + \frac{M_{grav}(r)\,G}{c^2\,R(r,t)}, \tag{17}$$

$$\pm t + t_R(r) = \int \frac{\mathrm{d}\tilde{R}}{\sqrt{c^2\left(f^2(r)-1\right) + \frac{2\,M_{grav}(r)\,G}{\tilde{R}}}}, \tag{18}$$

where $t_R(r)$ is the third undetermined function of the models, the bang time. In a similar manner, we can denote by $t_{R'}$ the time corresponding to layer intersections.

The TB models are defined up to some transformation of the co-moving coordinate $\psi : r \to \tilde{r}$, which decreases the number of undetermined functions from 3 to 2 (Just (1960) and Just & Kraus (1962)). These two undetermined functions should be chosen from the set:

$$t_R, \quad t_{R'}, \quad \rho_0, \quad f, \quad R_0, \quad \dot{R}_0. \tag{19}$$

If these functions are $t_R(r)$ and $t_{R'}(r)$, then the models are reduced to boundary problem for the equation of motion; in all other cases the models are reduced to the Cauchy problem. The transformation is not unique and may be choosen in accordance with the specific character of the problem to be solved. Different transformations are compared in Gromov (1996).

The transformation ψ is time independent, so it can be used to fix one of one of functions from the set (19). In this case one told about a parametrization of the TB model. Let us represent two examples.

Often (see, for instance Ribeiro (1992a, 1992b,1993), Liu (1990a, 1990b,1991), Gonçalves & Moss (1998)) the following transformation is used:

$$r = R(r, 0). \tag{20}$$

This implies that the function $F(r)$ is defined by the initial density profile $\rho(r, 0)$. This approach was fully studied by Liu (1990a, 1990b and 1991). An alternative transformation is based on the equality

$$r = M_{inv}(r)\,\frac{G}{c^2}. \tag{21}$$

It was first used in (Eardley 1974) and studied by Gromov (1996,1997, 1999). In (Gromov 1999) it is shown that the motivation for choosing (21) is that in this case the equation for the density (12), evaluated at the moment of the initial conditions, becomes an identity, and the TB models are reduced to the Cauchy problem for the equation of motion (9) with initial conditions (10) - (11). One advantage of this approach is that only equation (12) is needed to solve the dynamical problem.

The bang time is used as one of the initial conditions in the papers by Silk & Wilson (1979a), Olson & Silk (1979), Olson & Stricland (1990). They study the use of the TB models via a $3 + 1$ approach. Another way to approach this problem is to use observational coordinates.

2.2. TB models in observational cordinates

Observational coordinates are defined as being the set of coordinates $\{w, y, \theta, \phi\}$, where the set $\{w = const\}$ are the past light cones of the observer, y is the distance from the observer along a specific light cone, and (θ, ϕ) denote the coordinates of the object on the observer's "celestial sphere". In terms of these coordinates, the TB metric can be written as

$$\mathrm{d}s^2 = A^2\left(w,y\right)\mathrm{d}w^2 - 2A\left(w,y\right)B\left(w,y\right)\mathrm{d}w\mathrm{d}y - C^2\left(w,y\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

The Einstein field equations cannot be integrated explicitly in these coordinates, but the unknown functions can be related to observational parameters. As was shown by Ellis *et al* (1975), if $\{w = w_0\}$ is the past light cone of observation, then the unknown function $C\left(w,y\right)$ can be determined by

$$C\left(w_0, y(z)\right) = R(r(z), T(r(z)))$$

where R is defined by (1) and $t = T\left(r\right)$ is the equation of a past light ray. The function $B\left(w,y\right)$ can be found by calculating the total number of sources within a distance y from the observer

$$N\left(y\right) = 4\pi\int\limits_0^y n\left(w_0,x\right)B\left(w_0,x\right)C^2\left(w_0,x\right)\mathrm{d}x,$$

where $n\left(w_0,y\right)$ is the number density of sources. Lastly, the unknown function $A\left(w,y\right)$ can be eliminated by using the coordinate freedom of y on the light cone to set $A\left(w,y\right) = B\left(w,y\right)$. Thus, the problem reduces to 1) determining the equation of the past light ray, 2) relating this equation to the redshift, z, and 3) measuring the number count, $N\left(y\right)$, after either assuming or determining the form of $n\left(w_0,y\right)$.

A detailed study of this approach was made by first by Bonnor (1972). He did not use the past light cone directly, but instead considered initial conditions defined on a $t = const$ hypersurface. Full observational coordinates were considered by Ribeiro (1992a, 1992b, 1993), and later by Humphreys *et al* (1998b), Matravers (1998).

For our study, the dynamics requires the using of co-moving coordinate as independent radial coordinate. In this paper we study only initial conditions, so the independent radial coordinate may be choosen as R at $t = t_R$. The transformation between different forms of initial conditions is based on the first and second integrals of the equation of motion (9).

3. Domain of definition for Tolman-Bondi models with arbitrary bang time

We study the Tolman-Bondi models using two different sets of initial conditions:
A) bang time $t_R(R)$ and initial density profile $\rho_0(R)$
and
B) initial density $\rho_0(R)$ and velocity $\dot{R}_0(R)$ profiles.

Case A can be justified by a simple analogy. Consider an apple dropping from an apple tree. If the initial velocity, initial position and an equation of motion are given, we can calculate the time at which it will reach the ground. For the TB models a similar situation exists. If we start with $t_R(R)$ and $\rho_0(R)$ as initial conditions, we can calculate the velocity profile $\dot{R}(R_0)$ for $t = 0$. However, since a particle must arrive at the center by the time $t_R(R_0)$, it must have a predefined velocity at $t = 0$. In addition, since the density profile is also given, the gravitational potential becomes fixed by the same initial conditions. In general these two initial conditions, bang time $t_R(R)$ and initial density profile $\rho_0(R)$, are not compatable for all $R \geq 0$, but, probably, only for $R \geq R_*$.

3.1. Dimensionless equations

Before proceeding further, we restate the models in terms of dimensionless quantities. We use the following characteristic values:

$$l_0 = c\, t_0, \qquad t_0 = \frac{1}{H_0}, \qquad \Omega_0 = \frac{\rho(\infty)}{\rho_{cr}}, \qquad 8\,\pi\,\rho_{cr} = 3\,H_0^2 \qquad M_0 = \frac{4\,\pi}{3}\,\rho(\infty)\,l_0^3,$$

and dimensionless variables:

$$\xi = \frac{R}{l_0}, \qquad \tau = \frac{t}{t_0}, \qquad \delta(\xi) = \frac{\rho(R)}{\rho(\infty)}, \qquad \mu(\xi) = \frac{M_{grav}}{M_0} = 3\int\limits_0^{\xi} \delta(x)\,x^2\,\mathrm{d}x,$$

where l_0 is the characteristic length, t_0 is the characteristic time, Ω_0 is the density parameter of the FRW background, H_0 is the value of the Hubble parameter evaluated at the same moment as the initial conditions, $\rho(R)$ is the dust density, ρ_{cr} is the critical density, M_0 is the characteristic mass; ξ is the Euler radial coordinate, $\delta(\xi)$ is the dimensionless density, and $\mu(\xi)$ is the dimensionless mass M_{grav} of the dust. In terms of these quantities, the bang time can be written as $\tau_\xi(\xi)$. The index ξ reminds us that the bang time is the time required for the particle to come from its initial position ξ to $\xi = 0$.

The assumption of a unique bang time is often used. It is an essential assumption in studies by Olson & Silk (1979), as well as Teerikorpi *et al* (1992) and Ekholm & Teerikorpi (1993). In this section we show how one can use the first and the second integrals, equations (17) and (18), to restrict the domain of definition for TB models if we relax the assumption of simultaneous bang time.

We will use an effective ADM mass μ^*

$$\mu^* = \Omega_0\,\mu. \tag{22}$$

In terms of dimensionless variables the first integral of the equation of motion (17) becomes

$$\dot{\xi}^2(\mu,\tau) = f^2(\xi) - 1 + \frac{\mu^*}{\xi(\mu,\tau)}. \tag{23}$$

The second integral (18) has a different form depending on the sign of $f^2(\xi)-1$. For $f^2(\xi)-1<0$ (closed models):

$$\pm\tau+\tau_\xi(\xi)=\frac{\mu^*(\xi)}{(1-f^2(\xi))^{3/2}}\left(\arcsin\sqrt{\frac{1-f^2(\xi)}{\mu^*(\xi)}\,\xi(\tau)}-\right.$$
$$\left.\sqrt{\frac{1-f^2(\xi)}{\mu^*(\xi)}\,\xi(\tau)}\sqrt{1-\frac{1-f^2(\xi))}{\mu^*(\xi)}\,\xi(\tau)}\right)\,; \tag{24}$$

for $f^2(\xi)-1=0$ (flat models):

$$\xi^{3/2}=\xi_0^{3/2}\pm\frac{3}{2}\,\tau\,\sqrt{\mu^*}; \tag{25}$$

and for $f^2(\xi)-1>0$ (open models):

$$\pm\tau+\tau_\xi(\xi)=\frac{\mu^*(\xi)}{(f^2(\xi)-1)^{3/2}}\left(-\mathrm{arcsinh}\sqrt{\frac{f^2(\xi)-1}{\mu^*(\xi)}\,\xi(\tau)}+\right.$$
$$\left.\sqrt{\frac{f^2(\xi)-1}{\mu^*(\xi)}\,\xi(\tau)}\sqrt{1+\frac{f^2(\xi))-1}{\mu^*(\xi)}\,\xi(\tau)}\right)\,; \tag{26}$$

In the appendix we show the correspondence between the dimensional form used in astronomical literature and the dimensionsles form of the equations.

3.2. The closed and open models with arbitrary bang time

We are now ready to consider open and closed TB models with arbitrary bang time. We show in this section that closed model have an additional propertie, which produces nontrivial domain of definition of the TB models. By solving (23) for $1-f^2$ and substiuting this into (24), the expression for bang time τ_ξ of the closed and open models may be rewritten in the form:

$$\tau_\xi(\xi)=\sqrt{\frac{\xi^3}{\mu^*(\xi)}}\,\Psi(B), \tag{27}$$

where

$$B(\xi)=\frac{\xi\dot{\xi}^2(\xi)}{\mu^*(\xi)}\geq 0, \tag{28}$$

and for closed models

$$\Psi(B)\equiv\Psi^{cl}(B)=\frac{\arcsin\sqrt{1-B}-\sqrt{1-B}\,\sqrt{B}}{(1-B)^{3/2}},\qquad 0\leq B<1; \tag{29}$$

while for open models

$$\Psi(B)\equiv\Psi^{op}(B)=\frac{-\mathrm{arcsinh}\sqrt{B-1}+\sqrt{B}\,\sqrt{B-1}}{(B-1)^{3/2}},\qquad B>1. \tag{30}$$

The definition (28) also allows us to rewrite equation (23) as

$$B = (f^2 - 1)\,\frac{\xi}{\mu^*} + 1. \tag{31}$$

It follows from (28) that

$$B = 0 \tag{32}$$

corresponds to the following set of initial conditions: if $\xi = 0$, grad$\delta(\xi = 0) = 0$, when

$$\lim_{\xi\to 0} B(\xi) \sim \lim_{\xi\to 0}\left(\frac{\dot{\xi}}{\xi}\right)^2 \geq 0; \tag{33}$$

in case of $\xi \neq 0$, (32) implies

$$\dot{\xi} = 0. \tag{34}$$

This means that the physical cause of why a partical cannot come to an area $0 < \xi < \xi_{TB}$ being managed by the TB model with a given bang time and density profile as initial conditions is that the velocity of the particle is equal to zero at the boundary ξ_{TB}, see Figs. 2, 3, 4. For both cases

$$f^2 = 1 - \frac{\mu^*}{\xi} \geq 0 \qquad \text{for} \qquad B = 0, \tag{35}$$

which implies the nonequality

$$\xi \geq \mu^* \qquad \text{for} \qquad B = 0. \tag{36}$$

Note, that (35) restricts a kind of particular TB model in which the nonequality may be satisfied: because $f^2 \geq 0$ it follows from (35) that $f^2 - 1 < 0$. So, (32) may be satisfied only in the closed model.

The limit $B \to 1$ corresponds to $f \to 1$, so that both the open and closed models have a common limit which coincides with the flat model:

$$\Psi^{fl}(B) = \frac{2}{3}. \tag{37}$$

Olson & Silk (1979) defined the boundary between open and closed TB models with a simultaneous bang time as a place where $f = 1$. Here we show that in the case of an arbitrary bang time , and for a special class of initial conditions, there is also a second boundary for the closed models. To prove this, we assume the existence of a set of initial conditions for the models (for example, the fractal density and Hubble law), which produces the following sequence of particular models: a closed model which has a position around a center ("core") and open model that is farther out from the center ("shell"). The two models are separated by the flat model located on the surface where $f^2(\xi) = 1$. For the closed "core"

$$0 \leq B < 1, \tag{38}$$

so, from (29) and (31) it follows that

$$\frac{2}{3} < \Psi^{cl}(B) \leq \arcsin(1) \approx 1.57, \tag{39}$$

where $\frac{2}{3}$ corresponds to the well known boundary of the closed model, the flat model, (this boundary we will denote by ξ_{fl}) and arcsin(1) corresponds to the second (new) boundary which we are studing (this boundary we will denote by ξ_{TB}).

We now apply these results to the problem of formulating the domain of definition of TB models with the initial conditions described earlier and with a known, non constant bang time $\tau_\xi(\xi)$. The domain of definition has a form of nonequality

$$\xi > \xi_{TB}, \tag{40}$$

where ξ_{TB} is the solution of equation

$$\tau_\xi(\xi) = \sqrt{\frac{\xi^3}{\mu^*}} \arcsin(1). \tag{41}$$

The solution of equation (41) may be real or complex depending on initial conditions, i.e. bang time and density profile. If the solution is complex, this means that the domain of definition is trivial, $\xi \geq 0$. If the solution is real (and positive) this means that the domain of definition is $\xi \geq \xi_{TB}$ and the initial conditions are not compatible for $0 \leq \xi < \xi_{TB}$. But the initial density profile is defined for $\xi \geq 0$. In the domain $0 \leq \xi < \xi_{TB}$ we can introduce some TB model, also closed, but with another bang time. In reference (Humphreys at al 1998a) is it represented how to constract the TB model for that domain.

The above approach utilizes the coordinate's form of the criterium for the existence of a central domain in which no TB model is represented. Using (27), we can also define a second form for this criterium, the mass criterium. From (27) it follows that the two limits of function $\Psi(B)$ correspond to two characteristic masses. The mass $\mu_{TB}(\xi)$,

$$\mu_{TB}(\xi) = \left(\frac{1.57}{\tau_\xi(\xi)}\right)^2 \xi^3, \tag{42}$$

corresponds to the low limit of radial Euler coordinate ξ_{TB}. This denotes a starting point from which all particles can collapse at time $\tau_\xi(\xi)$. Similarly, the characteristic mass corresponding to the flat model (or to the upper boundary of the closed model, which is the same thing) has the form

$$\mu_{fl}(\xi) = \left(\frac{2}{3\,\tau_\xi(\xi)}\right)^2 \xi^3. \tag{43}$$

This criterium can be stated as follows: if the graph of the mass, corresponding to a given initial density profile, intersects the graph of μ_{TB}, then $\xi_{TB} > 0$.

3.3. The flat Tolman-Bondi model

We now turn to the simplest case of initial conditions, $f = 1$. If $\tau_\xi(\xi) = const$, the flat TB model reduces to the flat FRW model. As was shown by Gromov (1997), in the case of the flat TB model the bang time may be represented in the form:

$$\tau_\xi(\mu) = \sqrt{\frac{1}{\mu^*}\int\limits_0^\mu \frac{\mathrm{d}y}{\rho_0(y)}}, \tag{44}$$

which immediately implies that $\rho_0(\xi) = const.$ for simultaneous bang time. In any other case, $\rho_0(\xi) \neq const$ and the bang time is not constant. As we have shown earlier, ξ_{TB} may be not equal to zero (and the bang is not simultaneous) if and only if the TB model is closed, so the domain of definition of the flat TB model is the whole region $\xi \geq 0$.

The simultaneouse bang for any particular TB model time is separated by the request $\tau_\xi(\xi) = const.$

4. Tolman-Bondi models for a fractal density distribution with simultaneous and nonsimultaneous bang time

This section is devoted to the study of the TB models with initial conditions given by the fractal density profile and Hubble law.

4.1. Fractal density distribution and the Hubble law: Hubble-de Vaucouleurs paradox

Two fundamental empirical laws have been established from extragalactic data. First the power law density-distance relation (cosmological de Vaucouleurs law) which corresponds to fractal struture with fractal dimention $D \approx 2$ up to the depth of available catalogs, i.e. about 100 h^{-1} Mpc. (see the review by Silos Labini *et al* (1998)). The second is that observations of the Hubble law by means Cepheids, Tully-Fisher distance indicator and supernovae of Type Ia confirm the linearity of the redshift-distamnce relation within the same distance scales where the fractality exists. Deflections from linearity of the Hubble law are very small: peculiar velocities of about 60 - 70 km/sec are suggested for the general field (see Sandage (1995), Ekholm & Teerikorpi (1993)).

As it was emphasized by Baryshev *et al* (1998), the linearity of the redshift-distance relation inside the fractal (i.e. inhomogeneous) matter distribution creates the so-called Hubble-de Vaucouleur's (HdeV) paradox. It means that the interpretation of the Hubble law within FRW cosmological models as a consequence of homogeneity of the galaxy distribution is not compatible with the new data on spatial galaxy distribution.

Two possible solutions of the HdeV paradox have been proposed. The first one (Baryshev *et al* 1998) is based on the assumption of the existence of homogeniously distributed dark matter starting just from the halos of galaxies, in which case the standard FRW solution exists. However, then the fractal distribution of luminous matter (galaxies) can appear only from a special choice of small initial perturbation of FRW. †

The second solution is to accept a very low value for the global average density (see Baryshev *et al* (1998) and Humphreys *et al* (1998b). However in this case when the value of the upper cut off scale of the fractal structure is large, the low density contradicts the available estimates of the density of the barionic luminous and dark matter.

In this section we study another solution of the HdeV paradox. We show that with the nonsimultaneous bang time the linear Hubble law is compatible with a fractal structure having any upper cut off.

4.2. On the applicability of TB model to fractals

The first application of the TB model to the hierarchical cosmological models was done by Bonnor (1972), who used de Vaucouleurs density law $\rho \sim d^{-\gamma}$ with $\gamma = 1.7$. More recently Ribeiro (1992a, 1992b, 1993) in a series of papers developed a numerical approach to solving TB equation for fractal galaxy distribution. (Humpreys *et al* 1998b) gave an analytical relation between observed number counts and redshifts for TB models which have FRW behavior at large scales. In all these papers it is pointed out that to get the linear Hubble law one needs a very low value of the large scale FRW density.

However, in application of TB models to fractal density distribution there is a new conceptual problem which has been little discussed. This is the problem of the preferred centre point of density distribution. In original Lemaitre-Tolman-Bondi formulation it was suggested that there is a central point of the universe, around which the density distribution is spherically symmetric belongs to the structure has spherical symmetry (in average) matter distribution.

This isotropy of matter distribution around every structure point makes possible the application of TB models as an exact general relativistic cosmological model in which expansion of space becomes scale dependent.For Friedmann models the velocity of space expansion at distance "d" is determined by the mass of the sphere around every point.For TB models the space expansion at distance "d" is also managed by the mass of the sphere around each point of the fractal structure.

4.3. Simultaneous bang time

We have shown in section 3.3 that a simultaneous bang time and constant density imply the open FRW models. Here, using nonlinear TB models, we study a local density perturbation with arbitrary amplitude on the FRW background and demonstrate

† But, as it was shown by de Vega *et al* (1998), self gravitating (via Newtonian gravity) N-body systems have a quasi-equlibrium state which is fractal in its structure with a fractal dimension of 2 or 1.5 . So, self gravity naturally leads to fractality.

how the initial fractal density changes the models. The fractal density on the FRW background and simultaneous (FRW) bang are given by the initial conditions of the TB models:

$$\delta(\xi, 0) = \frac{A}{\epsilon + \xi} + 1, \tag{45}$$

$$\tau_\xi(\xi) = \tau_\xi(\infty) = \tau_{FRW} = const. =$$
$$\frac{1}{(1-\Omega_0)^{3/2}} \left(\sqrt{\frac{1-\Omega_0}{\Omega_0}} \sqrt{1 + \frac{1-\Omega_0}{\Omega_0}} - \arcsin \sqrt{\frac{1-\Omega_0}{\Omega_0}} \right), \tag{46}$$

here $\epsilon = \frac{R_{galaxy}}{l_0} \sim \frac{10\,Kpc}{5 \cdot 10^6\,Kpc} = 2 \cdot 10^{-6}$. Above the scale of a galaxy Eq.(45) describes the fractal density law with $D = 2$. The density contrast of the galaxy is $\delta(\xi = 0) \sim \frac{\rho_{galaxy}}{\rho_{background}} \sim \frac{10^{-25}\,g/cm^3}{10^{-29}\,g/cm^3} = 10^5$. So, we find that $A \sim 0.2$. For our calculations we use $A = 0.002, 0.02, 0.2, 2$, which imply the amplitude of the density $\delta(0) = 10^3, 10^4, 10^5, 10^6$, and we use $\Omega_0 = 0.001, 0.01, 0.1, 0.99$, which imply $\tau_{FRW} = 0.997, 0.98, 0.898, 0.688$.

The properties of the TB models with initial conditions (45), (46) are studied in the section 3. In this subsection we apply the results of section 3.2 to the initial conditions with given parameters. For choosen values of parameters A and Ω_0 TB models have a closed "core" and open "shell". But only for $\Omega = 0.001$ and $A = 0.002$ equation () 41 has a complex solution, see Fig. 1. This means that only these parameters produce the TB model with fractal density and simultaneous bang time with domain of definition $\xi \geq 0$ (see Table 4) where initial conditions are compatible. In all other cases of parameters Eq. () 34 has a real solution and the domain of definition is $\xi \geq \xi_{TB}$ (see Tables 1 - 4).

Figs. 2 and 3 represent the cases where coordinate and mass criteria for the existence of ξ_{TB} are applied for $A = 0.02$ and $\Omega = 0.01$.

Our solution depends on Ω_0: $\mu^* \sim \Omega_0$ and $\tau_{FRW} = \tau_{FRW}(\Omega_0)$. Tables 1 - 4 show characteristic values of ξ_{TB} and ξ_{fl} for different A and Ω_0. Here $l_0 = c/H_0 = 5000Mpc$, $H_0 = 60km^{-1} \cdot s^{-1} \cdot Mpc$.

At the end of this subsection we calculate the velocity $\dot{\xi}(\xi)$, produced by the initial conditions (45) and (46). For both domains of the model, closed and open,

$$\dot{\xi} = \sqrt{\frac{\mu^*}{\xi} B}, \tag{47}$$

where B is the solution of the equation (27). Fig.4 shows the resulting non-linear velocity-distance relations.

Tables A1-A4 and Fig.4 confirm the previous conclusion by Baryshev et al (1998) that the observed linear Hubble law is compatible with such a fractal density only if the FRW density parameter Ω_o is small. For instance, if $\Omega_o = 0.99$, then the zero-velocity radius R_{TB} ranges 3.4 - 3400 Mpc for the range of the density contrast $A = 0.002 - 2$, and is 344 Mpc for the "preferred" value of $A = 0.2$. With a very small value of Ω_o, 0.001, R_{TB} appears around 0.6 Mpc, which is an intergroup scale, while a good linear Hubble flow is reached around 6 Mpc.

4.4. Nonimultaneous bang time

In this subsection we study the initial conditions which follow from the observations in the domain from $1Mpc$ to $100Mpc$, i.e. the fractal density and the Hubble law:

$$\delta(\xi, 0) = \frac{A}{\epsilon + \xi} + 1, \tag{48}$$

$$\dot{\xi} = \xi. \tag{49}$$

The bang time in this situation is calculated by two formulas, depending on the closed and open domains of the model:

$$\tau_\xi^{cl}(\xi) = \frac{1}{(1 - B(\xi))^{3/2}} \left(-\sqrt{B(\xi)}\sqrt{1 - B(\xi)} + \arcsin\sqrt{1 - B(\xi)} \right), \tag{50}$$

$$\tau_\xi^{op}(\xi) = \frac{1}{(B(\xi) - 1)^{3/2}} \left(\sqrt{B(\xi)}\sqrt{B(\xi) - 1} - \operatorname{arcsinh}\sqrt{B(\xi) - 1} \right), \tag{51}$$

where

$$B(\xi) = \frac{\xi^3}{\mu^*(\xi)}, \tag{52}$$

follows from (26) and (49). For $A = 0.02$ and $\Omega_0 = 0.01$ the bang time $\tau_\xi(\xi)$ is shown in Fig.4.

5. Discussion and conclusion

We have discusse general case of assimptotically FRW cosmological models with simulataneouse and non simulataneouse bang time; found the low limit of domaine of definition of the TB models produced by arbitrary bang time.

It is shown that in the frame of the exact nonliner relativistic TB models it is possible to have a linear velocity - distance relation of the expanding space when matter distribution is fractal. This requires a non-unique bang time. The bang time $\tau(\xi)$ is calculated for the linear Hubble law and fractal matter distribution with fractal dimension $D = 2$.

Ω_0	ξ_{TB}	$R_{TB}(Mpc)$	ξ_{fl}	$R_{fl}(Mpc)$
0.001	0.001	6.3	0.007	33.6
0.01	0.012	61	0.066	330
0.1	0.1	530	0.66	$3.3\,10^3$
0.99	0.69	$3.4\,10^3$	450	$2\,10^6$

Table 1. $A = 2$, $\delta(0) = 10^6$.

Ω_0	ξ_{TB}	$R_{TB}(Mpc)$	ξ_{fl}	$R_{fl}(Mpc)$
0.001	$1.2\,10^{-4}$	0.61	$6.7\,10^{-4}$	3.3
0.01	0.001	6.1	0.007	33
0.1	0.01	53	0.06	332
0.99	0.07	344	45	$2\,10^5$

Table 2. $A = 0.2$, $\delta(0) = 10^5$.

Ω_0	ξ_{TB}	$R_{TB}(Mpc)$	ξ_{fl}	$R_{fl}(Mpc)$
0.001	$9\,10^{-6}$	0.05	$6.3\,10^{-5}$	0.32
0.01	$1.2\,10^{-4}$	0.6	$6.6\,10^{-4}$	3.3
0.1	0.001	5.3	0.007	33
0.99	0.007	34	4.5	$2\,10^4$

Table 3. $A = 0.02$, $\delta(0) = 10^4$.

Ω_0	ξ_{TB}	$R_{TB}(Mpc)$	ξ_{fl}	$R_{fl}(Mpc)$
0.001	*complex*	*complex*	$3.4\,10^{-6}$	0.017
0.01	$8.8\,10^{-6}$	0.04	$6.2\,10^{-5}$	0.31
0.1	10^{-4}	0.5	$6.6\,10^{-4}$	3.3
0.99	$6.8\,10^{-4}$	3.4	0.45	$2\,10^3$

Table 4. $A = 0.002$, $\delta(0) = 10^3$.

6. Acnovelegements

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Appendix A.

In this appendix we show how the dimensional flat ($f = 1$) solution of TB model is transformed to the dimensionless one. Dimensional equations for the flat model

$$R\,\dot{R} = 2\,G\,M. \tag{A1}$$

This has the solution

$$R^{3/2} = R_0^{3/2} \pm \frac{3}{\sqrt{2}}\sqrt{G\,M}\,t, \tag{A2}$$

where R_0 is the initial condition. The dimensionless form of the equation (A1) is:

$$\xi\,\dot{\xi} = \Omega_0\,\mu \tag{A3}$$

which has the solution

$$\xi^{3/2} = \xi_0^{3/2} \pm \frac{3}{2}\sqrt{\Omega_0\,\mu}\,\tau. \tag{A4}$$

So, when we go from the dimensional equation to the dimensionless one, then factor $3/\sqrt{2}$ transforms to $3/2$. This factor appears in the flat TB solutions, which are located on the sphere dividing the closed and open parts of the smooth TB solutions.

References

Baryshev Yu., Silos Labini F., Montuori M., Pietronero L. and Teerikorpi P. 1998 *Fractals* **6** No.3 p.231-243

Bondi H. 1947 *MNRAS* **107** 410-425.

Bonnor W. 1972 *MNRAS* **159** 261-268

T.Ekholm and P.Teerikorpi 1993 *A&A* **284** 369

Eardley D.M. 1974 *Phys.Rev.Lett.* **33** 442

Ellis G.F.R.W., Nel S.D., Maartens R., Stoeger W.R. and Whitman A.P. 1985 *Phys.Rep.* **124** 315-417.

Gromov A. 1996 *gr-qc/9612038.*

Gromov A. 1997 *gr-qc/9706013.*

Gromov A., Baryshev Yu.,Suson D., Teerikorpi P. 1999 in preparation
Humphreys N.P., Maartens R. and Matravers D. 1998a *gr-qc/9804023*
Humphreys N.P., Maartens R. and Matravers D. 1998b *gr-qc/9804025*
Just 1960 *Z.Astrophysik* **49** 19
Just and Kraus 1962 *Z.Astrophysik* **55** 127
Landau and Lifshits 1972 vol.2 p.386
Lemaitre G. 1933 *Ann.Soc.Sci Bruxelles* **A53** 51
Liu H.Y. 1990a *J.Math.Phys.* **31** 2462
Liu H.Y. 1990b *J.Math.Phys.* **31** 2459
Liu H.Y. 1991 *J.Math.Phys.* **32** 2279
Matravers D.R. 1998 *gr-qc/9808015*
Gonçalves S.M.C.V. and Moss I.G. *gr-qc 9702059*
Olson D.W.and Silk J. 1979 *ApJ.* 233 395-401
Olson D.W. and Stricland J.C. 1990 *ApJ.* **359** 263-266
Ribeiro M.B. 1992a *ApJ.* **388** 1
Ribeiro M.B. 1992b *ApJ.* **395** 29
Ribeiro M.B. 1993 *ApJ.* **415**
Silk J. and Wilson M.L. 1979a *ApJ.* **228** 641-646
Silk J. and Wilson M.L. 1979b *ApJ.* **233** 769-774
Silos Labini F., Montuori M. and Pietronero L. 1998 *Phys.Rep.* **293** 61-226
Sandage A., Tammann G. and Hardy H. 1972 *ApJ.* 172 **172** 253-263
Sandage A. 1995 in *The Deep Universe* eds. B.Binggeli R.Buser Springer
Tolman R. 1934 *Proc.Nat.Acad.Sci.* **20** 169
Teerikorpi P., Botinelli L., Gouguenheim L. and Paturel G. 1992 *A&A* **260** 17 - 20
de Vaucouleurs G. 1970 *Science* **167** 1203-1213
de Vega H., Sanchez N. and Combes F. 1998 *AhJ.* **500** 8

Figure 1. This figure illustrates how area of definition of TB models with simultaneous bang time depends on initial conditions. ξ_{TB}, the solution of equation (41), depends on initial conditions (45) and (46). The upper curve is the graf of the function $\tau_{FRW}\left(\frac{\xi^3}{\mu^*}\right)^{1/2}$ corresponds to $\Omega_0 = 0.001$ and $A = 0.02$ (see Table 3), since the low curve corresponds to $\Omega_0 = 0.001$ and $A = 0.002$ (see Table 4). The upper line is $\max\log(\Psi^{\rm cl}) = \log(\arcsin(1)) = 0.196$. The low line is $\min\log(\Psi^{\rm cl}) = \log(\frac{2}{3}) = -0.176$. A dimensionless galaxy scale is 10^{-6}. ξ_{TB} is a coordinate of intersection of a curve with upper line. It is shown that parameters $\Omega_0 = 0.001$ and $A = 0.02$ produce the intersection at the scale more that galaxy scale, what corresponds to real (and positive) solution of the equation (41), but the parameters $\Omega_0 = 0.001$ and $A = 0.002$ do not produce it. In the last case the solution of the equation (41) is complex.

Figure 2. In this figure the coordinate's form of criterium of existence of ξ_{TB} is represented for simultaneous bang time $\tau_{FRW} = 0.98$, $\Omega_0 = 0.01$ and $A = 0.02$ (see Table 3). The model is defined for $\xi \geq \xi_{TB}$. At the boundary ξ_{TB} velocity $\dot{\xi}(\xi_{TB}) = 0$, see Fig. 4.

Figure 3. In this figure the mass's form of criterium of existence of ξ_{TB} is represented for simultaneous bang time $\tau_{FRW} = 0.98$, $\Omega_0 = 0.01$ and $A = 0.02$ (see Table 3). The upper of two paralle lines corresponds to μ_{TB} and the low line corresponds to μ_{fl}, see equations (42) and (43). The model is defined for $\xi \geq \xi_{TB}$. At the boundary ξ_{TB} velocity $\dot{\xi}(\xi_{TB}) = 0$, see Fig. 4.

Figure 4. Velocity produced by initial conditions (45) and (46) (simultaneous (FRW) bang time and ginven density profile) for $\Omega_0 = 0.001, 0.01, 0.1$ and $A = 0.02$. Different Ω_0 produce different ξ_{TB}, in which $\dot{\xi}(\xi_{TB}) = 0$.

Figure 5. Here is shown the graph of the non-simultaneous bang time $\tau_\xi(\xi)$ produced by the initial conditions (48) and (49) with $\Omega_0 = 0.01$ and $A = 0.02$. At the infinity the time of collapse $\tau_\xi(\xi) \to \tau_{FRW} = 0.98$.